International Release of Mathematics and Mathematical Sciences

Available at http://irmms.org

Effect of Magnetic field on Buoyancy Driven Convection with Insulating Permeable Boundaries

A. K. Gupta[#1], S. K. Kalta[2]

[#1]Professor, Department of Mathematics, Himachal Pradesh University Centre for Evening Studies, Shimla-171001, India

[2]Assistant Professor, Department of Applied Science & Humanities, Atal Bihari Vajpayee, Govt. Institute of Engineering and Technology, Pragatinagar, Shimla-171202, India

Email: [#1]guptakdr@gmail.com, [2]kalta.sanjay@gmail.com

Abstract

The effect of uniform magnetic field acting vertically upward on the onset of steady buoyancy driven convection in a horizontal layer of liquid is investigated, using the classical linear stability analysis. Both the lower and upper boundary surfaces of the liquid layer are considered to be insulating and permeable. The Galerkin method is used to obtain the eigenvalue equation which is then computed numerically. It is observed that the limiting cases of the permeable parameters include various combinations of the hydrodynamic boundary conditions as special cases, in the presence of the magnetic field. Results of this analysis indicate that the uniform magnetic field has the stabilizing effect on the onset of convection. Further, the asymptotic behavior of the critical Rayleigh number for large values of the Chandrasekhar number is also obtained.

Keywords: Buoyancy, Convection, Insulating, Linear stability, Magnetic field.

1. Introduction

Stimulated by Bénard's [1, 2] experiments, Rayleigh [3] studied the dynamic origin of the onset of thermal convection in a horizontal layer of liquid and laid down the theoretical foundation of buoyancy driven convection. Rayleigh's theory was extended and generalized by Jeffreys [4], Low [5] and Pellew and Southwell [6]. Further, Chandrasekhar [7] studied the onset of thermal convection in an electrically conducting fluid layer for conducting case of the hydrodynamic boundary conditions to include the effect of uniform vertical magnetic field and established that magnetic field has stabilizing effect on the onset of stationary convection. Nakagawa [8] carried out experimental investigations of the problem in the presence of magnetic field. For a detail study and recent developments in this regard, one may be referred to the excellent research monographs, articles or review articles, notably by Drazin and Reid [9], Banerjee and Gupta [10] and Nield [11]. Recently, the effect of uniform vertical magnetic field on the onset of convection driven by both buoyancy and surface tension has been studied by Gupta and Dhiman [12] for thermally conducting case, and by Gupta and Surya [13] for thermally insulating case of the lower rigid boundary.

In this paper, we investigate the effect of uniform magnetic field on the onset of buoyancy driven thermal convection in the more general framework of the insulating permeable boundary conditions, using the classical linear stability analysis. The present analysis extends the work of Gupta and Kalta [14] to include the effect of uniform vertical magnetic field in an electrically conducting layer of liquid. The Galerkin method is used to obtain the eigenvalue equation analytically. The numerical results obtained for a wide range of the permeable boundary parameters and prescribed

Chandrasekhar number Q are presented. It is observed that the limiting cases of the permeable parameters include various combinations of the hydrodynamic boundary conditions namely, when both boundaries are dynamically free and when both boundaries are rigid and when either one is dynamically free while other one is rigid, as special cases in the presence of magnetic field. The results of this analysis indicate that the uniform vertical magnetic field has stabilizing effect on the onset of buoyancy driven convection in the more general framework of hydrodynamic boundary conditions. It is interesting to note that the critical wave number on the onset of convection is found to be zero. The asymptotic behavior of the critical Rayleigh number for large values of the Chandrasekhar number is also obtained.

2. Formulation of the Problem

We consider an infinite horizontal layer of viscous, incompressible and electrically conducting fluid of uniform thickness d heated from below, in the presence of uniform magnetic field $\vec{H}$ acting opposite to gravity $\vec{g}$, whose both boundary surfaces are insulating and permeable. We choose a Cartesian coordinate system with x and y axes in the plane of the lower boundary and positive direction of the z axis along the vertically upward direction so that the fluid layer is confined between the planes at $z=0$ and $z=d$. A uniform temperature gradient is maintained across the layer by maintaining the lower boundary surface at a uniform temperature T_0 and the upper one at temperature T_1. Following the usual procedure for obtaining the linearized perturbation equations (Chandrasekhar [15]), the non-dimensional form of the governing equations are given as

$$(D^2-a^2)(D^2-a^2-p)w+Q(\mathrm{D}^2-\mathrm{a}^2)Dh_z=Ra^2\theta, \tag{2.1}$$

$$(D^2-a^2-p\,P_r)\theta=-w, \tag{2.2}$$

$$(D^2-a^2-p\,P_m)h_z=-Dw. \tag{2.3}$$

where w is the z-component of the perturbation velocity, θ is the temperature perturbation, h_z is the z-component of the perturbation from the uniform vertical magnetic field $\vec{H}$, a is the horizontal wave number, $P_r\ (=\nu/\kappa)$ is the thermal Prandtl number, $P_m\ (=\nu/\eta)$ is the magnetic Prandtl number, $Q\,(=\mu_e H^2 d^4/4\pi\rho\eta\nu)$ is the Chandrasekhar number, $R\,(=g\alpha\beta d^4/\kappa\nu)$ is the Rayleigh number, α is the volume coefficient of thermal expansion, $\beta=(T_0-T_1)/d$ is the maintained temperature gradient, g is the gravitational acceleration, ν is the kinematic viscosity, κ is the thermal diffusivity, η is the magnetic resistivity, μ_e is the magnetic permeability, $p=p_r+ip_i$ represents the growth rate of perturbations (a complex constant in general), as p_r and p_i are real constants, and $D=d/dz$. We have chosen d, d^2/ν, ν/d and $\beta d\nu/\kappa$ as the units of length, time, velocity and temperature respectively.

Since both the lower and upper boundary planes are fixed, thermally insulating and electrically conducting, the associated boundary conditions are

$$w=0,\ D\theta=0 \text{ and } h_z=0 \quad \text{at } z=0 \text{ and } z=1. \tag{2.4}$$

Further, as both upper and lower boundary surfaces of the liquid layer are considered to be permeable on which the boundary condition as specified by Beavers and Joseph [16] is applicable. As described by Gupta et al. [17], the appropriate permeable boundary conditions are given by

$$D^2w-K_0\,Dw=0, \text{ at } z=0, \tag{2.5}$$

$D^2w + K_1 Dw = 0$, at $z = 1$. (2.6)

Where K_0 and K_1 are non-negative dimensionless parameters, characterizing the permeable nature of the lower and upper boundary respectively.

Eqns. (2.1) - (2.3) together with boundary conditions (2.4) - (2.6) pose an eigenvalue problem. We restrict our analysis to the case when the marginal state is stationary so that the marginal state is characterized by setting $p = 0$ in Eqns. (2.1) - (2.3) and h_z is eliminated from the resulting equations, we obtain

$$[(D^2 - a^2)^2 - \mathrm{Q}\mathrm{D}^2)]w = Ra^2\theta, \tag{2.7}$$

$$(D^2 - a^2)\theta = -w. \tag{2.8}$$

Eqn. (2.7) - (2.8) together with boundary conditions (2.4) - (2.6) constitute an eigenvalue problem of order six.

3. Solution of the Problem

The single term Galerkin method is convenient for solving the present problem (Finlayson [18]). Accordingly, the unknown variables w and θ are written as

$$w = Aw_1 \quad \text{and} \quad \theta = B\theta_1 \tag{3.1}$$

in which A and B are constants and w_1 and θ_1 are the trial functions, which are chosen suitably satisfying the boundary conditions (2.4) - (2.6). Multiplying Eqn. (2.7) by w and Eqn. (2.8) by θ, integrating the resulting equations with respect to z from 0 to 1 using the boundary conditions (2.4) - (2.6). Substituting for w and θ from (3.1) and eliminating A and B from resulting system of equations, we obtain the following system of linear homogeneous algebraic equations:

$$[K_1(Dw_1(1))^2 + K_0(Dw_1(0))^2 + \langle (D^2w_1)^2 + (2a^2 + \mathrm{Q})(Dw_1)^2 + a^4(w_1)^2 \rangle]A - Ra^2\langle w_1\theta_1\rangle B = 0, \tag{3.2}$$

$$\langle w_1\theta_1\rangle A - \langle (D\theta_1)^2 + a^2(\theta_1)^2\rangle B = 0. \tag{3.3}$$

The system of equations given by Eqns. (3.2) - (3.3) will have a non-trivial solution if and only if

$$R = \frac{1}{a^2\langle w_1\theta_1\rangle^2}[K_1(Dw_1(1))^2 + K_0(Dw_1(0))^2 + \langle (D^2w_1)^2 + (2a^2 + \mathrm{Q})(Dw_1)^2 + a^4(w_1)^2\rangle][\langle (D\theta_1)^2 + a^2(\theta_1)^2\rangle] \tag{3.4}$$

Where $\langle -- \rangle$ denotes integration with respect to z from z = 0 to z =1.

We select the trial functions satisfying the boundary conditions given by Eqns. (2.4) - (2.6) as

$$w_1 = z^4 - 2\frac{K_0K_1 + 5K_0 + 3K_1 + 12}{K_0K_1 + 4(K_0 + K_1) + 12}z^3 + \frac{K_0(K_1 + 6)}{K_0K_1 + 4(K_0 + K_1) + 12}z^2 + 2\frac{K_1 + 6}{K_0K_1 + 4(K_0 + K_1) + 12}z, \tag{3.5}$$

$$\theta_1 = 1. \tag{3.6}$$

Substitution of trial functions given by Eqns. (3.5) - (3.6) into the Eqn. (3.4) yields R in terms of a, K_0, K_1 and Q given by

$$R = \frac{10}{7\{K_0(K_1+9)+9(K_1+8)\}^2} \times$$

$$\big[504\{K_0(K_1+4)+4(K_1+3)\}\{K_0(K_1+9)+9(K_1+8)\} \tag{3.7}$$
$$+12(2a^2+\mathrm{Q})[72\{K_1(K_1+13)+51\}+3K_0\{5K_1(K_1+14)+312\}+K_0{}^2\{K_1(K_1+15)+72\}]$$
$$+a^4[76K_1(K_1+15)+K_0\{17K_1(K_1+16)+1140\}+K_0{}^2\{K_1(K_1+17)+76\}+4464]\big].$$

For given values of K_0, K_1 and Q, Eqn. (3.7) gives the Rayleigh number R as a function of wave number a. The minimum of R is the critical Rayleigh number R_c and the value of a at which R attains minimum is the critical wave number.

4. Numerical Results and Discussion

A close observation of the expression for R given by Eqn. (3.7) shows that R attains its minimum when $a = 0$ for any fixed values of K_0, K_1 and Q. We put $a = 0$ on the right hand side of the expression for R in Eq. (3.7) and obtain its minimum R_c as given by

$$R_c = 720\left[\frac{K_0K_1+4(K_0+K_1)+12}{K_0K_1+9(K_0+K_1)+72}\right]$$
$$+\frac{120}{7}\mathrm{Q}\left[\frac{72\{K_1(K_1+13)+51\}+3K_0\{5K_1(K_1+14)+312\}+K_0{}^2\{K_1(K_1+15)+72\}}{\{K_0(K_1+9)+9(K_1+8)\}^2}\right]. \tag{4.1}$$

When Q = 0, the expression given by Eq. (4.1) becomes identically same as that obtained by Gupta and Kalta [14] in absence of the magnetic field.

Table 1: Values of R_c for various values of K_0 and K_1 when Q = 0, 1, 10, 10^2 and 10^6.

		$Q=0$	$Q=1$	$Q=10$	$Q=10^2$	$Q=10^6$
K_0	K_1	R_c	R_c	R_c	R_c	R_c
0	0	120.00	132.14	241.43	1334.29	12.14×10^6
0	1	142.22	154.45	264.51	1365.03	12.23×10^6
0	10	231.11	244.33	363.27	1552.69	13.21×10^6
0	10^2	305.19	320.01	453.47	1788.10	14.83×10^6
0	10^6	320.00	335.24	472.38	1843.81	15.24×10^6
1	0	142.22	154.45	264.51	1365.03	12.23×10^6
1	1	166.15	178.43	288.89	1393.54	12.27×10^6
1	10	262.54	275.67	393.83	1575.43	13.13×10^6
1	10^2	343.68	358.36	490.49	1811.83	14.68×10^6
1	10^6	360.00	375.09	510.86	1868.57	15.09×10^6
10	0	231.11	244.33	363.27	1552.69	13.22×10^6
10	1	262.54	275.67	393.83	1575.43	13.13×10^6
10	10	392.73	406.26	528.03	1745.74	13.53×10^6
10	10^2	507.01	521.89	655.93	1996.31	14.89×10^6
10	10^6	530.53	545.82	683.44	2059.61	15.29×10^6
10^2	0	305.18	320.02	453.47	1788.10	14.83×10^6
10^2	1	343.68	358.36	490.49	1811.83	14.68×10^6
10^2	10	507.01	521.89	655.93	1996.31	14.89×10^6
10^2	10^2	655.72	671.98	818.37	2282.26	16.27×10^6
10^2	10^6	686.97	703.67	853.94	2356.67	16.70×10^6
10^6	0	320.00	335.24	472.38	1843.81	15.24×10^6
10^6	1	360.00	375.09	510.86	1868.57	15.09×10^6
10^6	10	530.53	545.82	683.44	2059.61	15.29×10^6
10^6	10^2	686.97	703.67	853.94	2356.67	16.70×10^6
10^6	10^6	720.00	737.14	891.43	2434.29	17.14×10^6

The numerical values of the critical Rayleigh number R_c, obtained using the Eqn. (4.1) with the aid of symbolic algebraic package Mathematica, for various assigned values of the parameters K_0, K_1 and Q, are presented in Table 1. From Table 1, we observe that an increase in the value of Q leads to an increased value of R_c indicating that the magnetic field has stabilizing effect on the onset of convection, for assigned values of the pair (K_0, K_1). On the other hand, for a prescribed value of Q, we observe that for a fixed value of either one of the parameters K_0 or K_1, an increase in the value of other one has stabilizing effect on the onset of convection.

Fig. 1 illustrates the variation of R_c with K_0 for various prescribed values of K_1 and Q. It clearly shows that increasing values of K_0 (or K_1) for a fixed value of Q has the stabilizing effect on the onset of convection. Also, Fig. 1 shows that an increase in the value of Q leads to an increased value of R_c, for a fixed value of K_1, indicating that the magnetic field strength has stabilizing effect on the onset of convection.

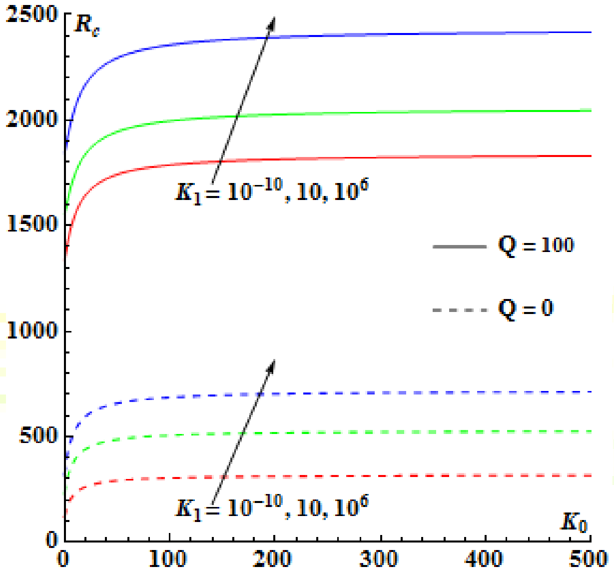

Figure 1: Variation of R_c with K_0 for various values of K_1 when Q = 0 and 100.

4.1 Limiting Cases

From Table 1, we observed that various limiting cases of the permeable parameters K_0 and K_1 give rise to different combinations of hydrodynamic boundary conditions namely, when both lower and upper boundaries are dynamically free that is, ($K_0 \to 0$ and $K_1 \to 0$) or when both lower and upper boundaries are rigid ($K_0 \to \infty$ and $K_1 \to \infty$) or when the lower boundary is free while upper one is rigid that is, ($K_0 \to 0$ and $K_1 \to \infty$) or when the lower boundary is rigid while upper one is free that is, ($K_0 \to \infty$ and $K_1 \to 0$), in presence of the magnetic field Q described as follows.

Case 1. When $K_0 \to 0$ and $K_1 \to 0$, that is, when both boundaries are dynamically free (free-free). In this case, from Eq. (4.1), we find that

$$R_c = 120 + \frac{85}{7} Q. \tag{4.2}$$

We find numerically that the asymptotic behavior of the critical Rayleigh number crucially depends on Q. In this case $R_c \approx 12.14Q$ when $Q \to \infty$.

Case 2. When either $K_0 \to 0$ and $K_1 \to \infty$, that is, when the lower boundary is free while upper one is rigid (free-rigid) or when $K_0 \to \infty$ and $K_1 \to 0$, that is, when the lower boundary is rigid while upper one is free (rigid-free). In this case, from Eq. (4.1), we find that

$$R_c = 320 + \frac{320}{21} Q. \qquad (4.3)$$

The expression given by Eq. (4.3) is identically same as that obtained by Gupta and Surya [13] corresponding to the classical case of linear stability analysis. Further, we find numerically that the asymptotic behavior of the critical Rayleigh number crucially depends on Q. In this case $R_c \approx 15.24Q$ when $Q \to \infty$.

Case 3. when $K_0 \to \infty$ and $K_1 \to \infty$, that is, when both boundaries are rigid (rigid-rigid). In this case, from Eq. (4.1), we find that

$$R_c = 720 + \frac{120}{7} Q. \qquad (4.4)$$

We find numerically that the asymptotic behavior of the critical Rayleigh number crucially depends on Q. In this case $R_c \approx 17.14Q$ when $Q \to \infty$.

The stabilizing effect of the magnetic field Q on the onset of convection for various limiting cases of K_0 and K_1 described above, is shown in Fig. 2.

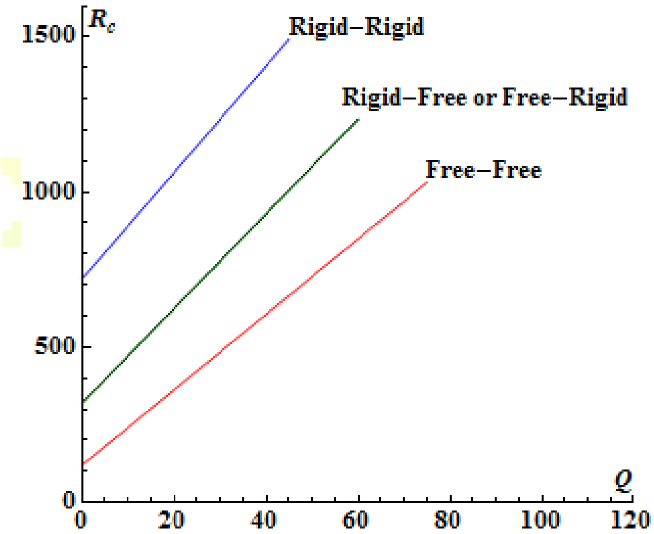

Figure 2: Variation of R_c as a function of Q for limiting cases of K_0 and K_1.

In addition, we consider a new theoretical case of possible practical interest which has not been discussed in literature so far despite its importance in problems related to science, engineering and technological fields. In this case, we consider that value of either of the two parameters characterizing the permeability varies inversely to that of the other in the presence of magnetic field. We let $K_1 = K_0^{-1}$, Eq. (4.1) then yields

$$R_c = 720\left[\frac{4K_0^2 + 13K_0 + 4}{9K_0^2 + 73K_0 + 9}\right] + \frac{120}{7} Q \left[\frac{72K_0^4 + 951K_0^3 + 3883K_0^2 + 951K_0 + 72}{(9K_0^2 + 73K_0 + 9)^2}\right] \qquad (4.5)$$

In Table 2, we have listed the numerical values of R_c for various values of Q and K_0. From Table 2, we observe that for a prescribed value of K_0, R_c increases with increase in Q, indicating the stabilizing effect of the magnetic field. Also, Table 2 shows that for a prescribed value of Q, as K_0 increases from 0 to ∞, R_c first decreases, attains its lowest minimum at $K_0 = 1$, and then increases. In other words, increasing value of the permeability parameter K_0 from 0 to 1 has destabilizing effect whereas increasing the value of K_0 from 1 onwards it has stabilizing effect on the onset of convection, in the presence of magnetic field.

Table 2: Values of R_c for various values of Q and K_0.

	$Q = 0$	$Q = 1$	$Q = 10$	$Q = 10^2$	$Q = 10^6$
K_0	R_c	R_c	R_c	R_c	R_c
10^{-10}	320.000	335.238	472.381	1843.810	15.24×10^6
10^{-1}	234.582	247.785	366.609	1554.850	13.20×10^6
1	166.154	178.428	288.893	1393.540	12.27×10^6
1.1	166.292	178.567	289.043	1393.800	12.28×10^6
10	234.582	247.785	366.609	1554.850	13.20×10^6
10^2	305.613	320.440	453.884	1788.330	14.83×10^6
10^6	319.998	335.236	472.379	1843.800	15.24×10^6

Further, asymptotically critical values of the Rayleigh number for large values of the Chandrasekhar number ($Q \to \infty$) are found to be $R_c > \pi^2 Q$ for any prescribed value of the parameter K_0 .

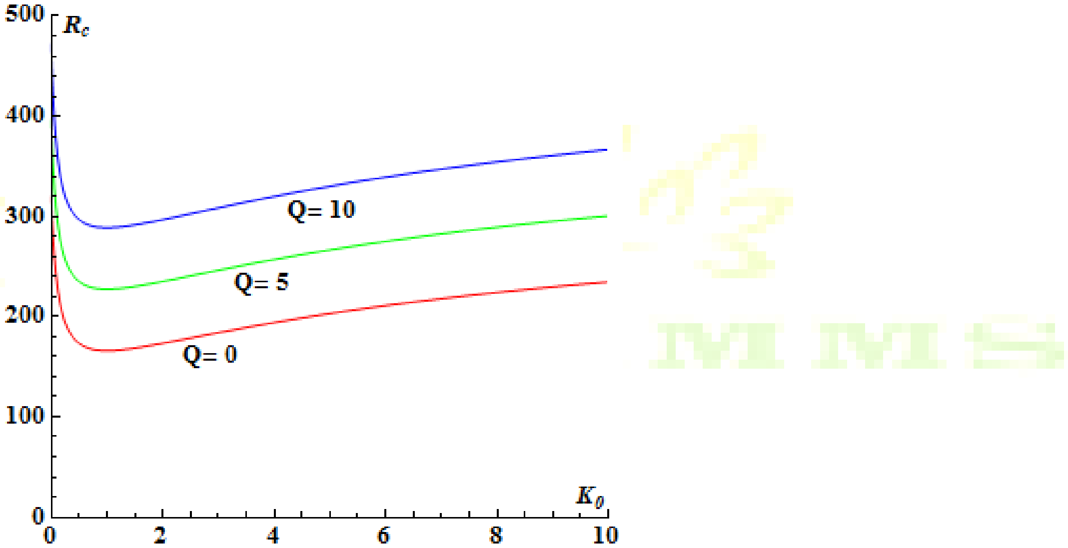

Figure 3: Variation of R_c with K_0 for various values of Q.

Fig. 3 illustrates the variation of R_c with K_0, for various values of Q. It clearly shows that the magnetic field strength has stabilizing effect on the onset of convection. Also, from Fig. 3, we observe that increasing values of K_0 from 0 to 1 has the destabilizing effect on the onset of convection and that the system becomes most unstable when $K_0 = 1$, for any prescribed value of Q.

5. Conclusions

The problem of onset of buoyancy driven thermal convection in a liquid layer heated from below in the presence of uniform vertical magnetic field with insulating permeable boundaries has been studied theoretically, using the classical linear stability analysis. We conclude that

1. The magnetic field strength always has stabilizing effect on the onset of convection in the more general framework of the hydrodynamic boundary conditions.

2. In presence of the magnetic field, for a fixed value of any one of the two permeability parameters K_0 or K_1, increasing values of the other one has stabilizing effect on the onset of convection. It is interesting to note that the critical wave number on the onset of convection is found to be zero.
3. For the case when $K_1 = K_0^{-1}$, increasing value of the permeability parameter K_0 from 0 to 1 has destabilizing effect whereas increasing the value of K_0 from 1 onwards it has stabilizing effect on the onset of convection, in the presence of magnetic field.
4. The asymptotic behavior of the critical Rayleigh number R_c obtained numerically for large values of the Chandrasekhar number is found to be $R_c > \pi^2 Q$, in the more general framework of thermal as well as hydrodynamic boundary conditions.

References

[1] H. Bénard, Les tourbillons cellulaires dans une nappe liquide, Rev. Gén. Sci. pures et Appl., Vol.11, 1900, pp. 1261-1271.

[2] H. Bénard, Les tourbillons cellulaires dans une nappe liquide transportant de la chaleur par convection en régime permanent, Ann. Chim. Phys, Vol. 23, 1901, 62-144.

[3] L. Rayleigh, On convection currents in a horizontal layer of fluid, when the higher temperature is on the underside, Phil. Mag. Vol. 32, 1916, pp. 529-546.

[4] H. Jeffreys, The stability of a layer of fluid heated below, Phil. Mag., Vol. 2, 1926, pp. 833-844.

[5] A. R. Low, On the criterion for stability of a layer of viscous fluid heated from below, Proc. Roy. Soc. London Ser. A, Vol. 125, 1929, pp. 180-195.

[6] A. Pellew and R. V. Southwell, On the maintained convective motion in a fluid heated from below, Proc. Roy. Soc. London Ser. A, Vol. 176, 1940, pp. 312-343.

[7] S. Chandrasekhar, On the inhibition of convection by a magnetic field, Phil. Mag., Vol. 43 (347), 1952, pp. 501-532.

[8] Y. Nakagawa, An experiment on the inhibition of thermal convection by a magnetic field, Nature, Vol. 175, 1955, PP. 417-419.

[9] P. G. Drazin and W. H. Reid, Hydrodynamic stability, Cambridge University Press, Cambridge, 1981.

[10] M. B. Banerjee and J. R. Gupta, Studies in hydrodynamic and hydromagnetic stability, Silver Line Publication, Shimla, 1991.

[11] D. A. Nield, The effect of permeable boundaries in the Bénard convection problem, J. Math. Phys. Sci., Vol. 26, 1992, pp. 341-343.

[12] A. K. Gupta and S. Dhiman, Effect of uniform magnetic field on Rayleigh-Bénard-Marangoni convection in a relatively hotter or cooler layer of liquid, Int. J. Adv. Res. Phy. Sci., Vol. 2(5) 2015, pp. 14-23.

[13] A. K. Gupta and D. Surya, Effect of magnetic field on Rayleigh-Bénard Marangoni convection in a relatively hotter or cooler layer of liquid with insulating boundaries, AJFAM, Vol. 5, 2017, pp. 1-9.

[14] A. K. Gupta S. K. Kalta, Buoyancy driven convection in a liquid layer with insulating permeable boundaries, AJFAM, Vol. 5, 2017, pp. 10-16.

[15] S. Chandrasekhar, Hydrodynamic and hydromagnetic stability, London, Oxford University Press, 1961.

[16] G. S. Beavers and D. D. Joseph, Boundary conditions at a naturally permeable wall, J. Fluid Mech., Vol. 30, 1967, pp.197-207.

[17] A. K. Gupta, R. G. Shandil and S. Kumar, On Rayleigh Bénard convection with porous boundaries, Proc. Natl. Acad. Sci. Sect. A, Phys. Sci. Vol. 83, 2013, pp. 365-369.

[18] B. A. Finlayson, The method of weighted residuals and variational principles, New York Academic Press, 1972.